Noise-level estimation of time series using coarse-grained entropy

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We present a method of noise-level estimation that is valid even for high noise levels. The method makes use of the functional dependence of coarse-grained correlation entropy $K_2(\varepsilon)$ on the threshold parameter ε. We show that the function $K_2(\varepsilon)$ depends, in a characteristic way, on the noise standard deviation σ. It follows that observing $K_2(\varepsilon)$ one can estimate the noise level σ. Although the theory has been developed for the Gaussian noise added to the observed variable we have checked numerically that the method is also valid for the uniform noise distribution and for the case of Langevin equation corresponding to the dynamical noise. We have verified the validity of our method by applying it to estimate the noise level in several chaotic systems and in the Chua electronic circuit contaminated by noise.

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I. INTRODUCTION

It is a common case that observed data are contaminated by a noise (for a review of methods of nonlinear time series analysis see [1,2]). The presence of noise can substantially affect invariant system parameters as a dimension, entropy or Lyapunov exponents. In fact Schreiber [3] has shown that even 2% of noise can make a dimension calculation misleading. It follows that the assessment of the noise level can be crucial for estimation of system invariant parameters. Even after performing a noise reduction one is interested to evaluate the noise level in the cleaned data. In the experiment the noise is often regarded as a measurement uncertainty which corresponds to a random variable added to the system temporary state or to the experimental outcome. This kind of noise is usually called the *measurement* or the *additive* noise. Another case is the noise influencing the system dynamics, what corresponds to the Langevin equation and can be called the *dynamical* noise. The second case is more difficult to analyze, because the noise acting at moment t_0 usually changes the trajectory for $t>t_0$. It follows that there is no clean trajectory and instead of it an ϵ-shadowed trajectory occurs [4]. For real data a signal (e.g., physical experiment data or economic data) is subjected to the mixture of both kinds of noise (measurement and dynamical).

Schreiber has developed a method of noise-level estimation [3] by evaluating the influence of noise on the correlation dimension of investigated system. The Schreiber method is valid for rather small Gaussian measurement noise and needs values of the embedding dimension *d*, the embedding delay τ and the characteristic dimension *r* spanned by the system dynamics.

Diks [5] investigated properties of correlation integral with the Gaussian kernel in the presence of noise. The Diks method makes use of a fitting function for correlation integrals calculated from time series for different thresholds ε. The function depends on system variables K_2 (correlation entropy), D_2 (correlation dimension), σ (standard noise deviation), and a normalizing constant Φ. These four variables are estimated using the least squares fitting. The Diks method [6] is valid for a noise level up to 25% of signal variance and for various measurement noise distributions. The Diks method needs optimal values of the embedding dimension *d*, the embedding delay τ, and the maximal threshold ε_c.

Hsu *et al.* [7] developed a method of noise *reduction* and they used this method for noise-level estimation. The method explored the local-geometric-projection principle and is useful for various noise distributions but rather small noise levels. To use the method one needs to choose a number of neighboring points to be regarded, an appropriate number of iterations as well as optimal parameters values *d* and τ.

Oltmans *et al.* [8] considered influence of noise on the probability density function $f_n(\varepsilon)$ but they could take into account only a small measurement noise. They used a fit of $f_n(\varepsilon)$ to the corresponding function which was found for small ε. Their fitting function is similar to the probability density distribution that we receive from correlation integrals $1/N^2\mathcal{D}_n(\varepsilon)$. The method needs as input parameters values of *d*, τ, and ε_c.

Our method has its origin in recurrence plots (RPs) [9] and it uses RPs quantities to characterize the data. Recurrence plots were originally introduced by Eckmann [9] as a useful graphic way for data analysis. The plot is defined as a matrix $N\times N$, where a dot (i,j) is drawn when $\|\vec{y}_i-\vec{y}_j\|<\varepsilon$ (ε is a given threshold). By recurrence plots one can study data stationarity [10–12], as well as their recurrence and deterministic properties [13–15]. The approach was also applied for parameter optimizing [16] in the local projection method of noise reduction [17]. RPs can be easy to calculate characteristic system parameters like the correlation entropy [18], what will be performed in our case. Lines of black dots parallel to the main diagonal can appear in recurrence plots and their number can serve as a measure of determinism [10]. In our method we take into account a number of lines $\mathcal{D}_n$ of the length *n* or longer by the embedding dimension

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$d=1$. We use the fact that there is a straightforward relation between $\mathcal{D}_n$ and the correlation integral [18].

The crucial point of our method is fitting of a proper function to the estimated correlation entropy K_2. In fact similar considerations can be performed for Kolmogorov-Sinai entropy [19–21] K_1 using, for example, the approach given in Ref. [22], but in such a case a much larger number of data is needed, since the K_1 entropy is more sensitive to regions of the phase space with small values of invariant measure. The method is not too time consuming, e.g., a calculation of entropy for 100 various thresholds and $N=3000$ data points needed a few minutes [23]. Our method does not demand any input parameters like the embedding dimension d or the embedding delay τ. The minimal and maximal values of the threshold parameter ϵ can be automatically estimated. In all considerations we use the maximum norm to save the computation time and to perform analytic expansions. It is known that in the limit $\epsilon \to 0$ the behavior of invariant system parameters does not depend on the type of used norm. In our case features of coarse-grained entropy are considered and the value of the threshold parameter ε should be comparable to the noise level. It follows that one can not exclude that the type of applied norm affects the functional dependence of the coarse-grained entropy $K_2(\varepsilon)$ in the presence of noise of a large or medium value.

We stress here that our method is provided for a noise-level estimation. The method is not equivalent to noise filters that allow to extract an original nondisturbed signal from noisy time series [4,24,25].

II. ENTROPY ESTIMATION FOR A TIME SERIES IN THE NOISE ABSENCE

Let $\{x_i\}$, where $i=1,2,\ldots,N$, be a time series and $\vec{y}_i = \{x_i, x_{i+\tau}, \ldots, x_{i+(n-1)\tau}\}$ a corresponding n-dimensional vector constructed in the embedded space, where n is an embedding dimension and τ is an embedding delay. The correlation integral calculated in the embedded space $\vec{y}_i$ is

$$C^n(\varepsilon) = \frac{1}{N^2} \sum_i^N \sum_{j\neq i}^N \theta(\varepsilon - \|\vec{y}_i - \vec{y}_j\|), \tag{1}$$

where θ is the Heaviside step function. If $\| \ldots \|$ is the maximum norm, the correlation integral $C^n(\varepsilon)$ is proportional to the number $\mathcal{D}_n(\varepsilon)$ of lines of the length n or longer in the RP constructed from the data set $\{x_i\}$ [18]

$$\begin{aligned} C^n(\varepsilon) &= \frac{1}{N^2} \sum_i \sum_{j\neq i} \theta(\varepsilon - |x_i - x_j|)\, \theta(\varepsilon - |x_{i+\tau} \\ &\quad - x_{j+\tau}|) \cdots \theta(\varepsilon - |x_{i+(n-1)\tau} - x_{j+(n-1)\tau}|) \\ &= \frac{\mathcal{D}_n(\varepsilon)}{N^2}. \end{aligned} \tag{2}$$

The correlation entropy [26,27] can now be calculated as

$$K_2 = \lim_{\varepsilon\to 0} \lim_{n\to\infty} \ln \frac{\mathcal{D}_n(\varepsilon)}{\mathcal{D}_{n+1}(\varepsilon)} \approx -\frac{d \ln[\mathcal{D}_n(\varepsilon)]}{dn}. \tag{3}$$

We assume that Eq. (3) is approximately valid for $n \geqslant 2$ thus

$$\mathcal{D}_n = \mathcal{D}_2 e^{-(n-2)K_2}. \tag{4}$$

Let us introduce the following convention for lines counting: if there is a line of the length n then it includes one line of the length $n-1$, one line of the length $n-2$, etc. Using Eq. (4) one can easily find the average line length $\langle n \rangle$,

$$\begin{aligned} \langle n \rangle &= \frac{\sum_{n=2}^{\infty} (\mathcal{D}_n + \mathcal{D}_{n+2} - 2\mathcal{D}_{n+1}) n}{\sum_{n=2}^{\infty} (\mathcal{D}_n + \mathcal{D}_{n+2} - 2\mathcal{D}_{n+1})} \cong \frac{\sum_{n=2}^{\infty} n e^{-(n-2)K_2}}{\sum_{n=2}^{\infty} e^{-(n-2)K_2}} \\ &= \frac{2 - e^{-K_2}}{1 - e^{-K_2}}. \end{aligned} \tag{5}$$

The above formula neglects all lines of the length $n=1$. Now the entropy can be approximated as

$$K_2 \approx \ln \frac{\langle n \rangle - 1}{\langle n \rangle - 2}. \tag{6}$$

The relation between the entropy, dimension, and correlation integral is given by the well-known formula [28,29]

$$\lim_{n\to\infty} \lim_{\varepsilon\to 0} \ln \frac{1}{N^2} \mathcal{D}_n(\varepsilon) = D_2 \ln \varepsilon - n\tau K_2, \tag{7}$$

thus the logarithm of the correlation integral is a linear function of entropy K_2 and system dimension D_2. On the other hand the correlation dimension D_2 is independent of the embedding dimension d if the latter is large enough. We use this fact and in the following section we will estimate the noise effect on the dimension D_2 as well as on the length n of the line in RP where the line length corresponds to the embedding dimension. Finally, we will incorporate both effects into Eq. (7) to reproduce the complete influence of noise on the correlation integral.

III. INFLUENCE OF NOISE ON CORRELATION INTEGRAL

Let us modify the definition of $\mathcal{D}_n$ in such a way that the influence of noise on entropy can be analytically estimated. First we change Eq. (1) to the equivalent form

$$\mathcal{D}_n(\varepsilon) = \sum_i^N \sum_{i\neq j}^N \theta\left(\sum_{k=0}^{l} \theta(\varepsilon - |x_{i+k} - x_{j+k}|) - n \right), \tag{8}$$

where l is the length of the recurrence line beginning at the point (i,j). Equation (8) is valid provided that one assumes $\theta(0)=1$ for the Heaviside function. The function θ in Eq. (1) is called a kernel function [30], and it can be written in a general way as $\rho_\varepsilon(r)$. Now let us use the fact [30] that the kernel function can be replaced by any monotonically decreasing function $\rho_\varepsilon(r)$ with a bandwidth ε such that $\lim_{r\to 0} r^{-p}\rho_\varepsilon(r)=0$ for $\varepsilon>0$ and any $p\geqslant 0$. The bandwidth ε of the kernel function corresponds to the threshold ε. It follows that we can replace the inner $\theta(\varepsilon-r)$ function in Eq. (8) by a new linear continuous function

$$\theta(\varepsilon-r)\Rightarrow\rho_\varepsilon(r)=\begin{cases}\dfrac{\varepsilon-r}{\varepsilon}, & \text{for}\quad 0\leqslant r\leqslant\varepsilon\\ 0, & \text{for}\quad r>\varepsilon,\end{cases} \tag{9}$$

and simultaneously we lower the threshold in outer θ function by the constant $\beta=1/\sqrt{\pi}$. We have checked that other choices of β bring similar results. Now instead of Eq. (8) we have

$$\mathcal{D}_n'(\varepsilon)=\sum_i^N\sum_{i\neq j}^N\theta\left(\sum_{k=0}^n\frac{\varepsilon-|x_{i+k}-x_{j+k}|}{\varepsilon}-\beta n\right). \tag{10}$$

We use the above expression to calculate the mean line length $\langle n\rangle$. Practically the length of each line is calculated as the maximal value of the parameter n in Eq. (10) provided that the θ function equals to 1. Having $\langle n\rangle$ we calculate the system entropy K_2 using the Eq. (6).

Now let us consider the influence of uncorrelated Gaussian noise η_i added to the observed system variable x_i. Eq. (10) is replaced by the following approximation:

$$\mathcal{D}_n'(\varepsilon)=\sum_i^N\sum_{i\neq j}^N\theta\left(\sum_{k=0}^n\frac{\varepsilon-|x_{i+k}+\eta_{i+k}-x_{j+k}-\eta_{j+k}|}{\varepsilon}-\beta n\right),$$

$$\cong\sum_i^N\sum_{i\neq j}^N\theta\left(\sum_{k=0}^n\frac{\varepsilon-|x_{i+k}-x_{j+k}|}{\varepsilon}-n\frac{\sqrt{\alpha^2\varepsilon^2+2\sigma^2}-\alpha\varepsilon}{\varepsilon}-\beta n\right), \tag{11}$$

where σ is the standard noise deviation and α is a constant of order 1 that depends on the distribution of $|x_i-x_j|$. One can easily derive Eq. (11) assuming that $\sigma_x\approx\alpha\varepsilon$, where σ_x stands for a standard deviation of $|x_i-x_j|\in(0,\varepsilon)$. When the differences $|x_i-x_j|$ are uniformly distributed in the region $(0,\varepsilon)$, then $\alpha=1/\sqrt{3}$.

Comparing Eq. (11) to Eq. (8) and Eq. (10) we see that the effect of noise corresponds formally to the change

$$n\to n\left(1+\sqrt{\pi}\,\frac{\sqrt{\varepsilon^2/3+2\sigma^2}-\varepsilon/\sqrt{3}}{\varepsilon}\right). \tag{12}$$

Instead of the second part of lhs of Eq. (7) we have

$$-n\tau K_2\to -n\tau K_2(\varepsilon)\left(1+\sqrt{\pi}\,\frac{\sqrt{\varepsilon^2/3+2\sigma^2}-\varepsilon/\sqrt{3}}{\varepsilon}\right). \tag{13}$$

For a small noise $(\sigma\ll\epsilon)$ the last equation can be transformed to

$$-n\tau K_2\to -n\tau K_2(\varepsilon)\left(1+\sqrt{3\pi}\frac{\sigma^2}{\varepsilon^2}\right), \tag{14}$$

which is in agreement with the well-known result [31,32] for the noise entropy in the case of noise spectrum $S(\omega)\sim\omega^{-2}$,

$$K_{noisy}\sim\frac{1}{\varepsilon^2}. \tag{15}$$

Equation (13) expresses the influence of noise on the line length n. On the other hand Schreiber has shown [3] that the influence of noise can be described by the substitution in Eq. (7),

$$D_2\to\left[D_2+(n-r)g\left(\frac{\varepsilon}{2\sigma}\right)\right], \tag{16}$$

where

$$g(z)=\frac{2}{\sqrt{\pi}}\frac{ze^{-z^2}}{erf(z)}, \tag{17}$$

and the parameter r follows from the method of singular value decomposition used in Ref. [3].

Combining Eq. (7) with results (13) and (16) we get

$$\mathcal{D}_n(\varepsilon)\sim\varepsilon^{[D_2+(n-r)g(\varepsilon/2\sigma)]}\exp\left[-n\tau K_2(\varepsilon)\times\left(1+\sqrt{\pi}\,\frac{\sqrt{\varepsilon^2/3+2\sigma^2}-\varepsilon/\sqrt{3}}{\varepsilon}\right)\right], \tag{18}$$

where $K_2(\varepsilon)$ is the coarse-grained entropy of the clean signal. The explicit form of the function $K_2(\varepsilon)$ is unknown. A good fit that seems to be valid for several systems is

$$K_2(\varepsilon)=\kappa+b\ln(1-a\varepsilon), \tag{19}$$

where the constant κ corresponds to the correlation entropy while the second term describes the effect of the coarse graining. We stress here that the precise value of the latter function is not needed for our approach of noise-level estimation, because we are left with some free parameters. It follows that one can estimate the coarse-grained entropy of the signal with noise as

$$K_{noisy}(\varepsilon)=-\frac{d\ln[\mathcal{D}_n(\varepsilon)]}{dn}=-\frac{1}{\tau}g\left(\frac{\varepsilon}{2\sigma}\right)\ln\varepsilon+K_2(\varepsilon)\times\left(1+\sqrt{\pi}\,\frac{\sqrt{\frac{\varepsilon^2}{3}+2\sigma^2}-\frac{\varepsilon}{\sqrt{3}}}{\varepsilon}\right), \qquad (20)$$

where the function $g(.)$ corresponds to the influence of noise on the correlation dimension, while the second term can be split into the coarse-grained entropy of the clean signal $K_2(\varepsilon)$ and the linear increase of this entropy due to the presence of the external noise $\sqrt{\pi}[(\sqrt{\varepsilon^2/3+2\sigma^2}-\varepsilon/\sqrt{3})/\varepsilon]K_2(\varepsilon)$. To estimate the noise level σ one can use the above dependence of the correlation entropy $K_{noisy}(\varepsilon)$ as the function of the threshold ε. However, we have found that because of a peculiar behavior of $K_{noisy}(\varepsilon)$ it is more convenient to fit the function $K_{noisy}(\varepsilon)\varepsilon^p$ instead of $K_{noisy}(\varepsilon)$ to corresponding experimental data (p is a constant of order of 1, see the following section for discussion). It follows that we need to estimate five free parameters κ, σ, a, b, and c for the function

$$K_{noisy}(\varepsilon)\varepsilon^p=-c\varepsilon^p g\left(\frac{\varepsilon}{2\sigma}\right)\ln\varepsilon+[\kappa+b\ln(1-a\varepsilon)]\varepsilon^p\times\left(1+\sqrt{\pi}\,\frac{\sqrt{\varepsilon^2/3+2\sigma^2}-\varepsilon/\sqrt{3}}{\varepsilon}\right). \qquad (21)$$

The parameter c (c ranges typically from 0.5 to 0.7) has been introduced for a better agreement to numerical data. To fit the above function we have used Levenberg-Marquardt method [33]. We stress here that we do not need to assume any input value for the above coefficients, but they appear as a result of application of our method.

IV. NOISE-LEVEL ESTIMATION: EXAMPLES

In practice, all input parameters of the method can be default. The character of the method causes that the evaluation of the embedding dimension that usually appears in nonlinear time series analysis is not needed at all. Since in RP we consider lines of all lengths larger than 2, the embedding dimension applied here is practically the highest as possible for given time series.

The first point is to calculate the average line length $\langle n\rangle$ for a given threshold and then to find the corresponding entropy $K_2(\varepsilon)$ using formula (6). Having values of entropies for about 100 different thresholds, one should rescale the ε axis. In such a way different systems with different sizes of attractors can be compared. Practically, we do this by multiplying ε by some constant γ, such that $\varepsilon_{max}\gamma=\overline{\varepsilon}_{max}=0.7$ [ε_{max} has been chosen using the condition $K(\overline{\varepsilon}_{max})=0.015$]. After finding the noise level $\overline{\sigma}$ in the rescaled data, the corresponding noise of the original time series can be calculated as $\sigma=\overline{\sigma}/\gamma$.

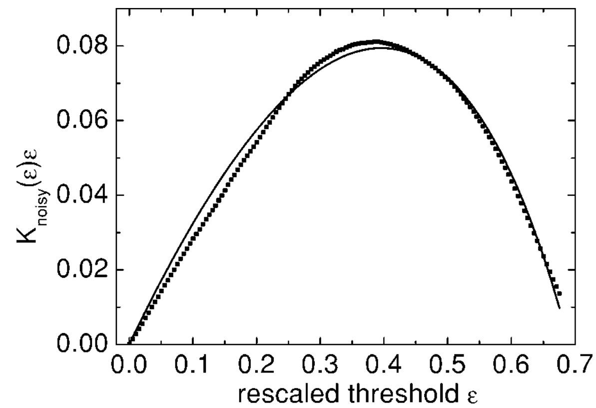

FIG. 1. Chaotic Henon map without a noise. Plot of coarse grained entropy multiplied by the threshold ε (squares) calculated from time series and the fitting function (21) with $p=1$ (line).

One can see the behavior of fitting function (21) for the clean signal in Fig. 1. For a small threshold $\varepsilon\ll\varepsilon_{max}$ the dependence is a linear since for small ε $K_2(\varepsilon)$ is a constant.

The important feature of the plot $K_{noisy}(\varepsilon)\varepsilon^p$ for noisy data is the appearance of two maxima (see Fig. 2). This feature is helpful for the noise estimation since origins of these maxima are related to the first and second part of rhs of Eq. (21), i.e., the first maximum is connected to the noise level, while the second maximum to the finiteness of the attractor. For a high noise level both maxima merge. The position of the first peak or the single maximum can be used for additional noise estimation, because one can find that for

$$p\simeq 3.441\,717-\frac{1}{\ln(\sigma)}, \qquad (22)$$

the maximum of $K_{noisy}(\varepsilon)\varepsilon^p$ appears at $\varepsilon=\sigma$. Relation (22) gives us the second way, beside Eq. (21), for estimation of noise level and for the control of results received due to fitting (21).

Let us define the percent of noise as the ratio of σ to the standard deviation of data

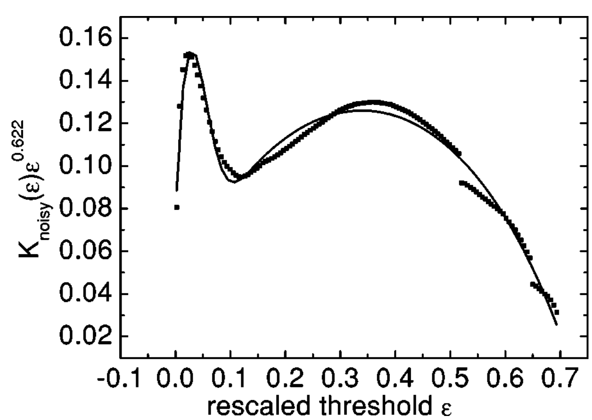

FIG. 2. Chaotic Henon map with measurement noise $\mathcal{N}\approx 10\%$. Plot of coarse-grained entropy calculated from the time series multiplied by $\varepsilon^{0.622}$ (squares) and fitting function (21) with $p=0.622$ (line).

TABLE I. Results of noise-level estimation for systems with the measurement noise.

System	$\mathcal{N}$	σ	Estimated σ
Henon	0%	0	-0.0023 ± 0.0001
Henon	9%	0.1	0.1 ± 0.0007
Duffing oscylator	20%	0.4	0.46 ± 0.005
Duffing oscylator	55%	2	1.9 ± 0.02
Ikeda	10%	0.07	0.07 ± 0.0005
Lorenz	22%	2.2	2.2 ± 0.01
Roessler	4%	0.58	0.58 ± 0.012
Roessler	14%	2	1.75 ± 0.01
Roessler	35%	6	6.16 ± 0.2
Roessler	48%	10	8.94 ± 0.1

$$\mathcal{N}=\frac{\sigma}{\sigma_{DATA}}\times100\% \tag{23}$$

The estimated values of the standard deviation σ received by an appropriate fit to Eq. (21) for several systems and noise levels are presented in Table I. One can see a fairly good agreement between the estimated and known level of noise.

We apply this method for chaotic differential equations where the noise $\vec{\eta}_n$ is added to system states $\vec{y}_n$, calculated by the fourth-order Runge-Kutta algorithm. It follows that next points of the trajectory are dependent in a nonlinear way on previous noisy contributions [34] (we call this kind of noise *a dynamical noise*). In fact we consider a noise added to the nonlinear map resulting from the original differential equations and the Runge-Kutta procedure $\vec{y}_{n+1}=F(\vec{y}_n+\vec{\eta}_n)$. We have found that the noise-level estimated by our method corresponds to the standard deviation of the noise existing in the system $\sigma=\sqrt{\langle\eta_n^2\rangle}$. Figure 3 shows that the behavior of the coarse-grained entropy is similar in the presence of dynamical and additive noise.

Results for the dynamical noise and a mixture of two kinds of noise are presented in Tables II and III. In Table II the first three examples correspond to the noise added after writing the value of a variable into a file and the next examples correspond to the noise added just before writing a variable to a file.

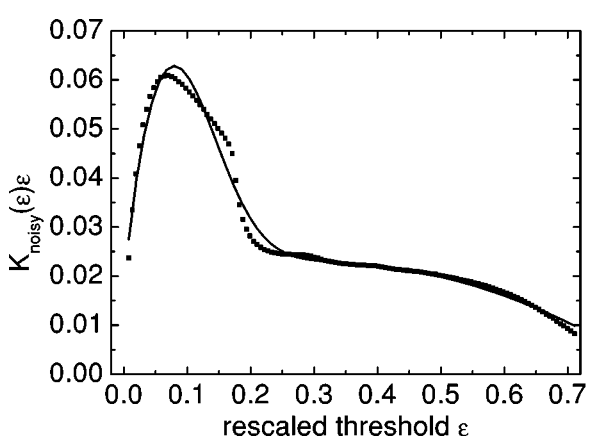

FIG. 3. Chaotic Lorenz model with the dynamical noise. Plot of coarse-grained entropy calculated from the time series multiplied by the threshold ε (squares) and fitting function (21) with $p=1$ (line).

TABLE II. Results of noise-level estimation for the Lorenz system with the dynamical noise.

System	$\mathcal{N}$	σ	Estimated σ
Lorenz	11%	1	1.19 ± 0.12
Lorenz	11%	1	1.17 ± 0.15
Lorenz	11%	1	1.14 ± 0.1
Lorenz	11%	1	1.15 ± 0.2
Lorenz	11%	1	1.11 ± 0.18
Lorenz	11%	1	1.09 ± 0.14

Our method can be useful for evaluation of very high noise levels. Figure 4 shows the plot of function (21) for the noise ($\mathcal{N}\approx100\%$, $p=1$). In such a case the error of the estimation is large, because we are free to use five parameters to fit a simple curve. We have found that for high noise levels it is better to use as the fitting function a sum of equations (21) with different exponents p (we have used $p_1=0.5$ and $p_2=7$). It follows that we fit the function $K_{noisy}(\varepsilon)(\varepsilon^{p_1}+\varepsilon^{p_2})$. The estimation works better, because for different values of p function (21) is more sensitive to different noise levels.

To verify our method in a real experiment we have performed analysis of data generated by a nonlinear electronic circuit. The Chua circuit in the chaotic regime [35,36] has been used, and we have added a measurement noise to the outcoming signal. The noise (white and Gaussian) has come from an electronic noise generator. The results are presented in Table IV. The first two rows correspond to $N=10\,000$ and the rest to $N=1000$. In the case of a small noise level we cannot perform any estimation for a small number of data, because the noise is smaller than the average distance between nearest neighbors. The estimation for $N=1000$ has taken a few minutes [23].

V. CONCLUSIONS

In conclusion we have developed a universal method of noise-level estimation from time series. The method makes use of the functional dependence of the coarse-grained entropy $K_2(\varepsilon)$ on the threshold ε. It appears that the peculiar

TABLE III. Results of noise-level estimation for systems with mixture of measurement and dynamical noise.

System	$\mathcal{N}$	σ	Estimated σ
Lorenz	43%	4.06	4.56 ± 0.12
Lorenz	56%	5.93	5.34 ± 0.11
Lorenz	35%	2.93	2.42 ± 0.12
Roessler	14%	2.82	1.97 ± 0.12
Roessler	94%	33.5	32 ± 0.75
Roessler	81%	16.12	16 ± 0.71

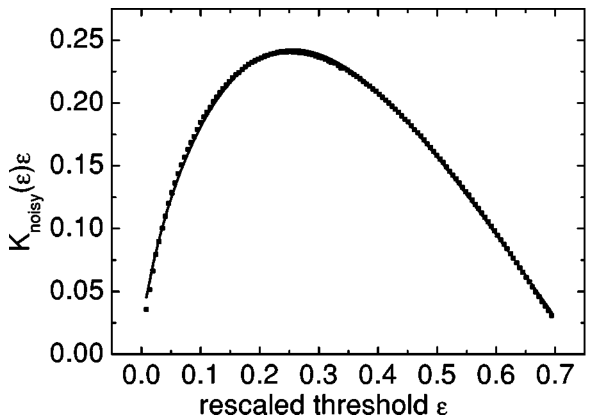

FIG. 4. Noise $\mathcal{N}\approx 100\%$. Plot of coarse-grained entropy calculated from the time series multiplied by the threshold ε (squares) and the fitting function (21) with $p=1$ (line).

shape of this entropy $K_2(\varepsilon)$ depends on the standard deviation of the noise σ so a simple function fitting can be applied to find the noise level. The process of noise estimation can be done easily without assuming input parameters and can be programmed in such a way that the algorithm makes all steps automatically. When the length of the time series $N<5000$ the whole evaluation procedure takes a few minutes [23]. The method has no limitations regarding a noise level and a kind of noise so one can evaluate very high noise levels and a dynamical noise as well. We have verified the validity of our method by applying it to estimate the noise level in several chaotic systems and in the Chua electronic circuit.

TABLE IV. Results of noise-level estimation for the Chua circuit with the measurement noise.

$\mathcal{N}$	σ (mV)	Estimated σ (mV)
0%	0	0.15 ± 0.015
3.1%	30.4	29.6 ± 0.3
6.2%	60.8	61.3 ± 8
12.3%	121.7	116 ± 8
24.9%	243.4	223 ± 13
28.3%	304	380 ± 9
46.1%	486	499 ± 20
73.7%	973	1109 ± 52
90.6%	1520	1537 ± 17
96.5%	2120	2042 ± 38

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